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Block cluster theory of site percolation on the four- and five-dimensional ordinary hypercubic lattices

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Abstract. Site percolation on the ordinary hypercubic lattice in four and five dimensions is studied using block cluster theory. The exact analytic renormalisation group equations of block cluster theory are obtained by a computer algorithm. They are solved numerically and yield, for the susceptibility exponent, $\gamma = 1.277$ ($d = 4$), $\gamma = 1.031$ ($d = 5$), and for the correlation length exponent $\nu = 0.566$ ($d = 5$).

1. Introduction

A wide variety of methods have been employed to study the percolation transition. They include series expansions [1, 2], Monte Carlo simulations [3-5], integral equations [6], rigorous methods [7] and renormalisation group techniques.

Renormalisation group techniques have been applied to the percolation transition both in Fourier space, leading to the ε expansion [8-10], and in real space. Among the several real space renormalisation transformations [11-17], block cluster theory has yielded *analytic* renormalisation group equations and accurate values of the thermal exponent in arbitrary dimension of space smaller or equal to six [17, 18].

The previous renormalisation group equations of block cluster theory, which yielded the thermal exponent in arbitrary dimension, were obtained for a special decorated hypercubic lattice. In this paper we present results for both the thermal and field exponents on the ordinary hypercubic lattice in four and five dimensions.

2. Theory

For the percolation transition, the basic requirement of renormalisation group theory that the free energy be conserved amounts, as discussed in detail in [17], to the preservation of the topological structure of the clusters upon renormalisation. Block cluster theory correspondingly ensures that the topology of the clusters in small blocks reflects that of those in the infinite lattice at the percolation transition. The conservation of the free energy upon renormalisation is optimised at the level of unit blocks by block cluster theory [17, 19].

The renormalisation group transformation in block cluster theory consists of a two-parameter transformation, with parameters h and p, where p is the probability that a site is occupied. The action of an external field on the system can be described by a ghost site which is connected to each site of the system with a probability $1 - e^{-h}$.

Letting p_b be the probability that a block is occupied and $1-e^{-h_b}$ be the probability that a block is connected to the ghost site, the renormalisation transformation is

$$p_b = R_1(p, h) \tag{1}$$

$$1-e^{-h_b} = R_2(p, h). \tag{2}$$

The probability of occupation is transformed from the site system to the block system by requiring, in accordance with the requirement of optimal conservation of the free energy, that each d-dimensional unit block contains a site cluster that spans all directions, i.e. is percolative. Correlations between blocks are neglected.

The probability of connection to the exterior site is transformed from the site to the block system by finding clusters in the block which span at least $d-1$ directions and which are such that at least one site belonging either to the cluster or to its boundary is connected by an occupied bond to the external site. Determination of the field exponent therefore requires not only the counting of all d-percolating and $(d-1)$-percolating clusters, but also the exact enumeration of the nearest neighbours of all these clusters.

Various ways of performing this enumeration have been found. Previous work has relied upon classifying topological shapes which can fit in a cell [18]. This method is an improvement over a direct counting of cluster configurations, since each shape represents a multiple number of actual clusters. However, as the dimension of space increases, the number of shapes becomes very large, so that this task becomes extremely tedious. The practical limitation for this exact method was found to be at the dimension $d=4$ for the ordinary hypercubic lattice [18].

3. Method of solution

We developed another method of enumerating percolating clusters which classifies them according to the configuration of sites disconnected from the cluster, i.e. sites neither in the cluster nor nearest neighbours of the cluster. Rules were developed to count arrangements of occupied sites which exhibit connectivity and percolation according to block cluster theory. We found that the categorisation of clusters by configurations of disconnected sites greatly facilitated the enumeration of clusters. This method enabled us to rederive the critical exponent, ν, in dimension $d=4$ and to obtain parts of the fixed-point equations in $d=5$.

However, in dimension $d=5$, a complete analytic enumeration of percolating clusters is far too tedious, as was found in earlier work [18]. On a five-dimensional hypercubic lattice, there are over 4×10^9 possible arrangements of occupied sites on the block, each of which must be checked for occupation and probability of connection to the ghost site. Not only do the numbers of clusters to be counted pose difficulties, but the possibility of having two separate percolating clusters on the same hypercube causes further complications.

The difficulty of the problem at hand precluded us from using a counting method based on cluster topology, as the chance for human error was too large. Instead, a computer program was written to determine the exact size of the clusters and to rigorously check for percolation. The requirements of the program were twofold: (i) to find the cluster(s) for each specific configuration and (ii) to check whether the cluster(s) percolated in the sense of block cluster theory.

The 32 sites on the five-dimensional hypercube were represented by the 32 bits of an integer *4 variable, which allowed all the accounting to be done by utilising FORTRAN intrinsic functions of a VAX 750 computer.

For each configuration clusters were first found by choosing an initial occupied site and checking whether neighbouring sites were occupied. If one or more occupied neighbours were found, the cluster became the initial site plus occupied neighbours. The process was repeated until all possible percolative clusters were found. It was next checked whether the cluster(s) did span d or $d-1$ dimensions of the hypercube.

4. Results

4.1. The field exponent in $d=4$

The probability that a block is percolative in $d-1$ dimensions and connected to the ghost site must first be found [11-16]. Equation (2) is, for $d=4$,

$$1-\exp(-h_b)=\sum_{k=5}^{16}\sum_{j=0}^{16} P_k^j p^k(1-p)^{16-k}[1-\exp(-jh)] \tag{3}$$

where P_k^j is the number of configurations of occupied sites which contain k occupied sites and j disconnected sites, and which are percolative in $d-1$ dimensions. The exact values of P_j^k can be found in appendix 1, where the explicit form of the right-hand side of (3) is given. The fixed point (h^*, p^*) occurs at $h^*=0$, $p^*=0.3582$.

The renormalisation group equations, i.e. equation (3) and that corresponding to equation (1), which was derived in [18], must then be linearised about their fixed points:

$$\begin{pmatrix} p_b-p^* \\ h_b-h^* \end{pmatrix} = \mathscr{L}_R \begin{pmatrix} p-p^* \\ h-h^* \end{pmatrix} \tag{4}$$

where $\mathscr{L}_R$ is the linearised renormalisation group operator. The eigenvalues λ_p and λ_h of $\mathscr{L}_R$ determine the thermal exponent y and the field exponent x:

$$\lambda_p = L^y \tag{5}$$

$$\lambda_h = L^x \tag{6}$$

where L is the lattice rescaling factor, which equals 2 for the ordinary hypercubic lattice. From x and y, all critical exponents can be found, e.g.

$$\gamma=(2x-d)/y \tag{7}$$

and

$$\nu = 1/y. \tag{8}$$

4.2. The thermal and field exponents in $d=5$

Having determined the exact number of all percolating configurations in $d=5$, the fixed point p^* is the solution $p_b=p=p^*$ of equation (1) with $h=0$. The analytic form of (1) is given in appendix 2. The root of the polynomial $R_1(p,0)-p$, with R_1 given by the right-hand side of (A2.1), is found numerically to be

$$p^*=0.258\,484.$$

From equation (5) we evaluate

$$\lambda_p = 3.405\,67$$

giving, from equation (8), the thermal exponent:

$$\nu = 0.565\,63.$$

The analytic form of the function $R_2(p, h)$ of (2) is given in appendix 3. Equation (A3.1) of appendix 3 yields the field eigenvalue and exponent

$$\lambda_h = 10.6194$$

and

$$\gamma = (2x - d)/y = 1.0305$$

respectively.

5. Conclusion

The value of the correlation length exponent ν, obtained by an exact enumeration of all percolative blocks in $d = 5$, compares very favourably with values obtained from the ε expansion, Monte Carlo techniques and block cluster theory on a decorated lattice for both site and bond percolation (see table 1), while the values of the field exponent γ compare much less favourably with those obtained by the two former methods (see table 2). In other words, the block cluster theory optimisation of the conservation of the free energy preserves rather well the connectivity of clusters, accounting for the accurate values of the thermal exponent. On the other hand, it is more difficult to simulate the shape of large clusters in small blocks. This leads to an enhanced compactness of the clusters in the block system, amounting to low values of the field exponent and, consequently, of the susceptibility critical exponent γ, as seen in this study.

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Appendix 1

For $d = 4$, the analytic form of the function $R_2\ (p, h)$ of equation (2) is

$$\begin{aligned}R_2(p, h) = {} & (64p^4q^{12} + 336p^5q^{11} + 720p^6q^{10} + 800p^7q^9 + 224p^8q^8 + 16p^9q^7) \\ & \times [1 - \exp(-11h)] + (192p^4q^{12} + 768p^5q^{11} + 1152p^6q^{10} + 768p^7q^9) \\ & \times [1 - \exp(-12h)] + (576p^5q^{11} + 1824p^6q^{10} \\ & + 2016p^7q^9 + 864p^8q^8 + 96p^9q^7)[1 - \exp(-13h)] \\ & + (192p^5q^{11} + 1472p^6q^{10} + 2912p^7q^9 + 2256p^8q^8 + 704p^9q^7 + 80p^{10}q^6) \\ & \times [1 - \exp(-14h)] + (576p^6q^{10} + 2608p^7q^9 + 3872p^8q^8 \\ & + 2624p^9q^7 + 960p^{10}q^6 + 192p^{11}q^5 + 16p^{12}q^4)[1 - \exp(-15h)] + (192p^6q^{10} \\ & + 1728p^7q^9 + 5584p^8q^8 + 8000p^9q^7 + 6968p^{10}q^6 + 4176p^{11}q^5 + 1804p^{12}q^4 \\ & + 560p^{13}q^3 + 120p^{14}q^2 + 16p^{15}q + p^{16})[1 - \exp(-16h)]. \qquad \text{(A1.1)}\end{aligned}$$

Table 1. Correlation length exponent ν.

	d				
Method	2	3	4	5	6
Analytic solution of the block cluster theory					
Ordinary hypercubic lattice	1.39 [17]	0.88 [17]	0.68 [18]	0.566†	—
Decorated hypercubic lattice	1.39 [18]	0.86 [18]	0.65 [18]	0.54 [18]	0.475 [18]
Bond percolation	1.63 [18]	0.93 [18]	0.69 [18]	0.57 [18]	0.497 [18]
Other methods‡					
Series	1.34 [1]	0.88 [1]			
Monte Carlo	1.35 ± 0.01 [3]	0.84 ± 0.01 [3] 0.88 ± 0.05 [4] 0.878 ± 0.003 [5]	0.7 ± 0.1 [3] 0.68 ± 0.3[4]	0.6 ± 0.1 [3]	0.5 ± 0.1 [3]
ε expansion	1.07 [8, 9] 1.31 [10]	0.83 [8, 9] 0.89 [10]	0.68 [8, 9] 0.69 [10]	0.57 [8, 9] 0.57 [10]	0.50 [8, 9] 0.50 [10]
Other	$\frac{4}{3}$§	0.89 [16] 1.0 [6] 1.04 [14] 1.22 [11]	0.64 [15]	0.51 [15]	—

† This work.
‡ Ordinary hypercubic lattice only.
§ Conjectured exact value.

Table 2. Field exponent γ (ordinary hypercubic lattice).

	d				
Method	2	3	4	5	6
Block cluster theory	2.344 [17]	1.692 [17]	1.277†	1.031†	—
Series expansion	2.43 ± 0.03 [21] 2.42 ± 0.02 [22]	1.66 ± 0.07 [21] 1.66 ± 0.02 [22]	1.40 ± 0.02 [22] 1.48 ± 0.08 [23]	1.17 ± 0.02 [22] 1.18 ± 0.07 [23]	1.08 ± 0.02 [22] 1.04 ± 0.06 [23]
ε expansion	2.41 [8-10]	1.81 [8-10]	1.44 [8-10]	1.18 [8-10]	1.0 [8-10]
Monte Carlo	2.29 ± 0.01 [24]	1.6 ± 0.1 [25] 1.8 ± 0.05 [20] 1.78 ± 0.09 [4]	1.6 ± 0.1 [20] 1.43 ± 0.07 [4]	1.3 ± 0.1 [20]	1.0 ± 0.05 [20]
Other	2.405 [16]	1.79 [16]	1.43 [15]	1.21 [15]	

† This work.

Appendix 2

For $d = 5$, the analytic form of equation (1) with $h = 0$ is

$$\begin{aligned} p_b = {} & 6912p^6q^{26} + 138\,752p^7q^{25} + 1287\,360p^8q^{24} + 7331\,200p^9q^{23} + 28\,741\,776p^{10}q^{22} \\ & + 82\,678\,048p^{11}q^{21} + 182\,186\,896p^{12}q^{20} + 318\,837\,440p^{13}q^{19} \\ & + 459\,174\,640p^{14}q^{18} + 562\,580\,128p^{15}q^{17} + 600\,707\,928p^{16}q^{16} \\ & + 565\,722\,720p^{17}q^{15} + 471\,435\,600p^{18}q^{14} + 347\,373\,600p^{19}q^{13} \end{aligned}$$

$$+225\,792\,840p^{20}q^{12}+129\,024\,480p^{21}q^{11}+64\,512\,240p^{22}q^{10}$$
$$+28\,048\,800p^{23}q^{9}+10\,518\,300p^{24}q^{8}+3365\,856p^{25}q^{7}+906\,192p^{26}q^{6}$$
$$+201\,376p^{27}q^{5}+35\,960p^{28}q^{4}+4960p^{29}q^{3}+496p^{30}q^{2}+32p^{31}q+p^{32} \qquad \text{(A2.1)}$$

where $q \equiv 1-p$.

Appendix 3

For $d=5$, the analytic form of the function $R_2(p,h)$ of equation (2) is

$$R_2(p,h)=(160p^{5}q^{27}+2592p^{6}q^{26}+19\,712p^{7}q^{25}+93\,440p^{8}q^{24}+309\,120p^{9}q^{23}$$
$$+704\,320p^{10}q^{22}+1068\,736p^{11}q^{21}+1056\,704p^{12}q^{20}+667\,360p^{13}q^{19}$$
$$+262\,720p^{14}q^{18}+70\,240p^{15}q^{17}+13\,408p^{16}q^{16}+1664p^{17}q^{15}$$
$$+96p^{18}q^{14})[1-\exp(-16h)]+(1920p^{5}q^{27}+26\,880p^{6}q^{26}+174\,720p^{7}q^{25}$$
$$+698\,880p^{8}q^{24}+1921\,920p^{9}q^{23}+3492\,480p^{10}q^{22}$$
$$+3964\,800p^{11}q^{21}+2676\,480p^{12}q^{20}+996\,480p^{13}q^{19}+176\,640p^{14}q^{18}$$
$$+13\,280p^{15}q^{17})[1-\exp(-18h)]+(1920p^{5}q^{27}+26\,560p^{6}q^{26}$$
$$+170\,880p^{7}q^{23}+678\,080p^{8}q^{24}+1855\,360p^{9}q^{23}+3485\,760p^{10}q^{22}$$
$$+4297\,280p^{11}q^{21}+3360\,960p^{12}q^{20}+1613\,120p^{13}q^{19}+457\,280p^{14}q^{18}$$
$$+80\,400p^{15}q^{17}+7680p^{16}q^{16}+320p^{17}q)[1-\exp(-19h)]$$
$$+(9120p^{6}q^{26}+109\,440p^{7}q^{25}+601\,920p^{8}q^{24}+2006\,400p^{9}q^{23}$$
$$+4514\,400p^{10}q^{22}+6423\,360p^{11}q^{21}+5382\,720p^{12}q^{20}+2521\,920p^{13}q^{19}$$
$$+600\,480p^{14}q^{18}+58\,560p^{15}q^{17}+1440p^{16}q^{16})[1-\exp(-20h)]$$
$$+(96\,000p^{6}q^{26}+116\,800p^{7}q^{25}+653\,230p^{8}q^{24}+2221\,120p^{9}q^{23}$$
$$+5121\,600p^{10}q^{22}+7898\,880p^{11}q^{21}+7704\,320p^{12}q^{20}+4603\,520p^{13}q^{19}$$
$$+1654\,400p^{14}q^{18}+340\,800p^{15}q^{17}+38\,080p^{16}q^{16}+1920p^{17}q^{15})$$
$$\times[1-\exp(-21h)]+(3840p^{6}q^{26}+67\,680p^{7}q^{25}+471\,360p^{8}q^{24}$$
$$+1838\,080p^{9}q^{23}+4600\,240p^{10}q^{22}+7803\,840p^{11}q^{21}+8243\,280p^{12}q^{20}$$
$$+5139\,840p^{13}q^{19}+1983\,680p^{14}q^{18}+524\,640p^{15}q^{17}$$
$$+95\,520p^{16}q^{16}+10\,400p^{17}q^{15}+400p^{18}q^{14})[1-\exp(-22h)]$$
$$+(43\,200p^{7}q^{25}+436\,960p^{8}q^{24}+1995\,680p^{9}q^{23}+5425\,920p^{10}q^{22}$$
$$+9742\,080p^{11}q^{21}+11\,333\,760p^{12}q^{20}+7640\,800p^{13}q^{19}+2777\,920p^{14}q^{18}$$
$$+549\,760p^{15}q^{17}+53\,920p^{16}q^{16}+1280p^{17}q^{15})[1-\exp(-23h)]$$
$$+(15\,360p^{7}q^{25}+264\,720p^{8}q^{24}+1632\,160p^{9}q^{23}+5390\,560p^{10}q^{22}$$
$$+11\,066\,240p^{11}q^{21}+14\,996\,320p^{12}q^{20}+12\,725\,120p^{13}q^{19}$$

$$
\begin{aligned}
&+6150\,560p^{14}q^{18}+1710\,080p^{15}q^{27}+313\,280p^{16}q^{16}\\
&+36\,320p^{17}q^{15}+1920p^{18}q^{14})[1-\exp(-24h)]\\
&+(5760p^{7}q^{25}+158\,400p^{8}q^{24}+1254\,400p^{9}q^{23}+4932\,960p^{10}q^{22}\\
&+11\,537\,760p^{11}q^{21}+17\,407\,840p^{12}q^{20}+17\,204\,000p^{13}q^{19}\\
&+10\,535\,200p^{14}q^{18}+3815\,360p^{15}q^{17}+894\,560p^{16}q^{16}\\
&+151\,680p^{17}q^{15}+15\,680p^{18}+q^{14}+640p^{19}q^{13})[1-\exp(-25h)]\\
&+(68\,640p^{8}q^{24}+846\,880p^{9}q^{23}+4240\,464p^{10}q^{22}+11\,698\,268p^{11}q^{21}\\
&+19\,962\,024p^{12}q^{20}+22\,111\,460p^{13}q^{19}+15\,831\,600p^{14}q^{18}\\
&+7029\,316p^{15}q^{17}+1973\,232p^{16}q^{16}+421\,408p^{17}q^{15}+80\,832p^{18}q^{14}\\
&+11\,840p^{19}q^{13}+992p^{20}q^{12}+32p^{21}q^{11})[1-\exp(-26h)]\\
&+(23\,040p^{8}q^{24}+510\,720p^{9}q^{23}+3475\,200p^{10}q^{22}+11\,845\,600p^{11}q^{21}\\
&+23\,878\,560p^{12}q^{20}+30\,577\,760p^{13}q^{19}+25\,503\,680p^{14}q^{18}\\
&+13\,708\,040p^{15}q^{17}+4567\,840p^{16}q^{16}+901\,600p^{17}q^{15}\\
&+97\,920p^{18}q^{14}+4640p^{19}q^{13})[1-\exp(-27h)]\\
&+(9600p^{8}q^{24}+2600\,160p^{9}q^{23}+2500\,480p^{10}q^{22}+11\,021\,280p^{11}q^{21}\\
&+27\,338\,200p^{12}q^{20}+42\,321\,960p^{13}q^{19}+43\,092\,000p^{14}q^{18}\\
&+29\,582\,800p^{15}q^{17}+13\,808\,480p^{16}q^{16}+4372\,520p^{17}q^{15}+926\,240p^{18}q^{14}\\
&+125\,760p^{19}q^{13}+9800p^{20}q^{12}+320p^{21}q^{11})[1-\exp(-28h)]\\
&+(97\,920p^{9}q^{23}+1551\,360p^{10}q^{22}+9172\,240p^{11}q^{21}+28\,453\,200p^{12}q^{20}\\
&+53\,734\,840p^{13}q^{19}+66\,866\,040p^{14}q^{18}+57\,401\,280p^{15}q^{17}\\
&+34\,936\,160p^{16}q^{16}+15\,298\,360p^{17}q^{15}+4830\,680p^{18}q^{14}\\
&+1082\,720p^{19}q^{13}+164\,480p^{20}q^{12}+15\,200p^{21}q^{11}+640p^{22}q^{10})\\
&\times[1-\exp(-29h)]+(19\,200p^{9}q^{23}+669\,792p^{1q22}+6044\,540p^{11}q^{21}\\
&+25\,322\,912p^{12}q^{20}+61\,518\,960p^{13}q^{19}+97\,248\,160p^{14}q^{18}\\
&+107\,249\,584p^{15}q^{17}+86\,363\,568p^{16}q^{16}+52\,311\,904p^{17}q^{15}\\
&+24\,238\,960p^{18}q^{14}+8626\,720p^{19}q^{13}+2335\,056p^{20}q^{12}\\
&+467\,552p^{21}q^{11}+65\,536p^{22}q^{10}+5760p^{23}q^{9}+240p^{24}q^{8})\\
&\times[1-\exp(-30h)]+(159\,360p^{10}q^{22}+2527\,800p^{11}q^{21}\\
&+15\,694\,240p^{12}q^{20}+52\,854\,720p^{13}q^{19}+112\,806\,140p^{14}q^{18}\\
&+167\,555\,188p^{15}q^{17}+184\,190\,056p^{16}q^{16}+156\,159\,288p^{17}q^{15}\\
&+104\,894\,520p^{18}q^{14}+56\,710\,200p^{19}q^{13}+24\,835\,968p^{20}q^{12}\\
&+8794\,016p^{21}q^{11}+2491\,808p^{22}q^{10}+553\,760p^{23}q^{9}+93\,280p^{24}q^{8}\\
&+11\,232p^{25}q^{7}+864p^{26}q^{6}+32p^{27}q^{5})[1-\exp(-31h)]\\
&+(42\,640p^{10}q^{22}+735\,660p^{11}q^{21}+5996\,456p^{12}q^{20}+2805\,9840p^{13}q^{19}\\
&+83\,914\,320p^{14}q^{18}+175\,148\,900p^{15}q^{17}+273\,457\,274p^{16}q^{16}\\
&+335\,955\,904p^{17}q^{15}+336\,330\,180p^{18}q^{14}+280\,808\,320p^{19}q^{13}\\
&+198\,446\,224p^{20}q^{12}+119\,747\,340p^{21}q^{11}+61\,954\,256p^{22}q^{10}\\
&+27\,489\,280p^{23}q^{9}+10\,424\,780p^{24}q^{8}+3354\,624p^{25}q^{7}+905\,328p^{26}q^{6}\\
&+201\,344p^{27}q^{5}+35\,960p^{28}q^{4}+4060p^{29}q^{3}+496p^{30}q^{2}+32p^{31}q+p^{32})\\
&\times[1-\exp(-32h)]. \qquad\qquad (A3.1)
\end{aligned}
$$

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